

LOWER BOUNDS FOR NODAL SETS OF DIRICHLET AND NEUMANN EIGENFUNCTIONS

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ABSTRACT. Let φ be a Dirichlet or Neumann eigenfunction of the Laplace-Beltrami operator on a compact Riemannian manifold with boundary. We prove lower bounds for the size of the nodal set $\{\varphi = 0\}$.

1. INTRODUCTION

Let (M, g) be a compact smooth Riemannian manifold with boundary. Let Δ be the Laplace-Beltrami operator. Let $\lambda \geq 1$ and let φ be an eigenfunction of $-\Delta$, i.e. a smooth real-valued function on M with

$$-\Delta\varphi = \lambda\varphi$$

over the interior of M . We will assume that φ is a Dirichlet eigenfunction, meaning

$$\varphi|_{\partial M} = 0$$

or a Neumann eigenfunction, meaning

$$\partial_\nu\varphi|_{\partial M} = 0$$

where ν is the outward unit normal vector on ∂M and ∂_ν is the corresponding directional derivative. Define the nodal set

$$Z = \left\{x \in M : \varphi(x) = 0, x \notin \partial M\right\}$$

Let n be the dimension of M and let \mathcal{H} be the $(n-1)$ -dimensional Hausdorff measure on M . We will prove lower bounds for $\mathcal{H}(Z)$.

We use the notation $A \lesssim B$ to mean there is a positive constant C , independent of λ and φ , such that $A \leq CB$.

Theorem 1.1. *If φ is a Neumann eigenfunction, then*

$$\lambda^{\frac{5-2n}{6}} \lesssim \mathcal{H}(Z)$$

If φ is a Dirichlet eigenfunction and $n \leq 3$, then

$$\lambda^{\frac{5-2n}{6}} \lesssim \mathcal{H}(Z)$$

If the boundary is strictly geodesically concave and φ is a Dirichlet eigenfunction, then for $n \leq 4$,

$$\lambda^{\frac{3-n}{4}} \lesssim \mathcal{H}(Z)$$

If (M, g) is a compact real analytic Riemannian manifold with boundary, then Donnelly and Fefferman [2] proved that

$$\lambda^{1/2} \lesssim \mathcal{H}(Z) \lesssim \lambda^{1/2}$$

If (M, g) is a compact smooth Riemannian manifold without boundary, then Colding and Minicozzi [1] proved that

$$(1.1) \quad \lambda^{\frac{3-n}{4}} \lesssim \mathcal{H}(Z)$$

This same result was later obtained by Hezari and Sogge [6]. Their argument was based on the identity

$$(1.2) \quad \lambda \int_M |\varphi| dV = 2 \int_Z |\nabla \varphi| dS$$

where dV is the Riemannian volume measure and dS is the Riemannian surface measure on Z . This identity had been proven by Sogge and Zelditch [10], who also showed that

$$(1.3) \quad \lambda^{-\frac{n-1}{8}} \lesssim \int_M |\varphi| dV$$

Hezari and Sogge [6] proved that

$$(1.4) \quad \int_Z |\nabla \varphi|^2 dS \lesssim \lambda^{3/2}$$

and then used (1.2), (1.3), and (1.4) to obtain the bound (1.1).

We will prove analogues of (1.2), (1.3), and (1.4) for a compact smooth Riemannian manifold with boundary. This will enable us to establish Theorem 1.1. In particular, we will prove the following.

Theorem 1.2. *If φ is a Dirichlet or Neumann eigenfunction, then*

$$\lambda \int_M |\varphi| dV = \int_{\partial M} |\partial_\nu \varphi| dS + 2 \int_Z |\nabla \varphi| dS$$

More generally, for any function f in $C^2(M)$,

$$\int_M ((\Delta + \lambda)f) |\varphi| dV = \int_{\partial M} f |\partial_\nu \varphi| dS + \int_{\partial M} |\varphi| \partial_\nu f dS + 2 \int_Z f |\nabla \varphi| dS$$

For a Neumann eigenfunction, the first term on the right side is zero, and this identity is the same as (1.2). For a Dirichlet eigenfunction, the integral over ∂M is an additional obstacle and causes the argument to break down in higher dimensions.

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2. PROOFS

Define

$$P = \left\{ x \in M : \varphi(x) > 0, x \notin \partial M \right\}$$

and

$$N = \left\{ x \in M : \varphi(x) < 0, x \notin \partial M \right\}$$

We can write M as a disjoint union

$$M = P \cup N \cup \partial M \cup Z$$

Define

$$\Omega = \left\{ x \in M : \varphi(x) = 0 \right\}$$

and

$$\Sigma = \left\{ x \in \Omega : \nabla \varphi(x) = 0 \right\}$$

Lemma 2.1. *If φ is a Dirichlet or Neumann eigenfunction, then $\mathcal{H}(\Omega) < \infty$, and the Hausdorff dimension of Σ is at most $n - 2$. If φ is a Neumann eigenfunction, then the Hausdorff dimension of $\Omega \cap \partial M$ is at most $n - 2$.*

Proof. Fix a point p in M . To prove the first statement, it suffices to find a neighborhood U of p in M such that $\mathcal{H}(\Omega \cap U) < \infty$ and $\Sigma \cap U$ has Hausdorff dimension at most $n - 2$. If $\varphi(p) \neq 0$, then finding such a neighborhood U is trivial. So we assume $\varphi(p) = 0$. By Donnelly and Fefferman [2], the eigenfunction φ only vanishes to finite order at p . If p is in the interior of M , we use geodesic normal coordinates about p . Then by Hardt and Simon [5], we can obtain U .

If p is on the boundary ∂M , then we use boundary normal coordinates (x_1, \dots, x_n) about p . These are defined by first letting (x_1, \dots, x_{n-1}) be geodesic normal coordinates on ∂M about p , with respect to the metric on ∂M induced by g . Then for fixed x_1, \dots, x_{n-1} , the curves $x_n \rightarrow (x_1, \dots, x_n)$, for $x_n \geq 0$, are geodesics in M which intersect ∂M normally. These coordinates are well-defined near p and allow us to identify some neighborhood of p with

$$B_+ = \left\{ x \in \mathbb{R}^n : |x| < \varepsilon, x_n \geq 0 \right\}$$

for some small $\varepsilon > 0$. Here the point p is being identified with the origin in \mathbb{R}^n . Let g_{ij} be the Riemannian metric on B_+ . Let

$$B = \left\{ x \in \mathbb{R}^n : |x| < \varepsilon \right\}$$

We extend the metric g_{ij} to B so that it is even in the x_n -variable. Let g^{ij} be the cometric, defined so that the matrix $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. Define

$$J = \left(\det[g_{ij}] \right)^{1/2}$$

The functions g_{ij} , g^{ij} , and J are Lipschitz continuous and bounded on B . If φ is a Dirichlet eigenfunction, extend φ to B so that it is odd in the x_n -variable. If φ is a Neumann eigenfunction, extend φ to B so that it is even in the x_n -variable. Then the extended function φ is in $C^1(B) \cap H^2(B)$. Let ψ be a smooth function on \mathbb{R}^2 with compact support contained strictly inside B . By Green's identity,

$$\sum_{i,j=1}^n \int_B (D_j \varphi)(D_i \psi) J g^{ij} dx = \int_B \lambda \varphi \psi J dx$$

That is,

$$\left(\sum_{i,j=1}^n D_i J g^{ij} D_j \varphi \right) + \lambda J \varphi = 0$$

We can write this equation as

$$\left(\sum_{i,j=1}^n J g^{ij} D_i D_j \varphi + (D_i J g^{ij}) D_j \varphi \right) + \lambda J \varphi = 0$$

Now by Hardt and Simon [5], we can obtain U .

It remains to prove the second statement. Fix a point p in $(\Omega \setminus \Sigma) \cap \partial M$. It suffices to show that there is a neighborhood V of p in ∂M such that the Hausdorff dimension of $(\Omega \setminus \Sigma) \cap V$ is at most $n - 2$. The set $\Omega \setminus \Sigma$ is a hypersurface with normal vector $\nabla \varphi(p)$ at p . Since φ is a Neumann eigenfunction and $\nabla \varphi(p) \neq 0$, the sets $\Omega \setminus \Sigma$ and ∂M intersect transversally, which yields V . \square

In particular, it follows that ∂P is smooth almost everywhere, with respect to \mathcal{H} , so the divergence theorem and Green's identities hold on P . See, e.g., Evans and Gariepy [3]. Let η be the outward unit normal on ∂P , defined at these smooth points, and let ∂_η be the corresponding directional derivative. On $Z \setminus \Sigma$, we have

$$\eta = -\frac{\nabla \varphi}{|\nabla \varphi|}$$

At any point on $\partial M \cap \partial P$ where η is defined, we have

$$\eta = \nu$$

Proof of Theorem 1.2. By Green's identity,

$$\begin{aligned} \int_P ((\Delta + \lambda)f)|\varphi| dV &= \int_P ((\Delta + \lambda)f)\varphi dV \\ &= \int_P f(\Delta + \lambda)\varphi dV - \int_{\partial P} f\partial_\eta \varphi dS + \int_{\partial P} \varphi \partial_\eta f dS \\ &= - \int_{\partial P \cap \partial M} f\partial_\eta \varphi dS - \int_Z f\partial_\eta \varphi dS + \int_{\partial P \cap \partial M} \varphi \partial_\eta f dS \\ &= \int_{\partial P \cap \partial M} f|\partial_\nu \varphi| dS + \int_Z f|\nabla \varphi| dS + \int_{\partial P \cap \partial M} |\varphi|\partial_\nu f dS \end{aligned}$$

The last equality holds because $-\partial_\eta \varphi = |\partial_\nu \varphi|$ over $\partial P \cap \partial M$ and $-\partial_\eta \varphi = |\nabla \varphi|$ over $\partial P \cap Z$. We can similarly obtain

$$\int_N ((\Delta + \lambda)f)|\varphi| dV = \int_{\partial N \cap \partial M} f|\partial_\nu \varphi| dS + \int_Z f|\nabla \varphi| dS + \int_{\partial N \cap \partial M} |\varphi|\partial_\nu f dS$$

Now

$$\begin{aligned} \int_M ((\Delta + \lambda)f)|\varphi| dV &= \int_P ((\Delta + \lambda)f)|\varphi| dV + \int_N ((\Delta + \lambda)f)|\varphi| dV \\ &= \int_{\partial M} f|\partial_\nu \varphi| dS + \int_{\partial M} |\varphi|\partial_\nu f dS + 2 \int_Z f|\nabla \varphi| dS \end{aligned}$$

\square

The following lemma is an analogue of (1.3).

Lemma 2.2. *If φ is a Dirichlet or a Neumann eigenfunction, then*

$$\lambda^{\frac{1-n}{6}} \lesssim \|\varphi\|_{L^1(M)}$$

If the boundary is strictly geodesically concave and φ is a Dirichlet eigenfunction, then

$$\lambda^{\frac{1-n}{8}} \lesssim \|\varphi\|_{L^1(M)}$$

Proof. Fix p satisfying $2 < p < \frac{2(n+1)}{n-1}$. Then, by Smith [7],

$$(2.1) \quad \|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(n-1)(p-2)}{6p}}$$

If the boundary is strictly geodesically concave and φ is a Dirichlet eigenfunction, then by Grieser [4] and Smith-Sogge [8],

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(n-1)(p-2)}{8p}}$$

Let $\theta = \frac{p-2}{2(p-1)}$. By Hölder's inequality,

$$1 = \|\varphi\|_{L^2(M)} \leq \|\varphi\|_{L^1(M)}^\theta \|\varphi\|_{L^p(M)}^{1-\theta}$$

The estimates now follow. \square

Remark. On the flat unit disc $\{|x| \leq 1\}$ in \mathbb{R}^2 , there are whispering gallery modes, which are concentrated near the boundary. It follows from Grieser [4] that Lemma 2.2 is sharp for these eigenfunctions. However, for $n \geq 3$, Smith and Sogge [9] conjectured that (2.1) can be strengthened to

$$(2.2) \quad \|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(3n-2)(p-2)}{24p}}$$

Applying Hölder's inequality as above would then yield

$$\lambda^{\frac{2-3n}{24}} \lesssim \|\varphi\|_{L^1(M)}$$

The following lemma is an analogue of (1.4).

Lemma 2.3. *If φ is a Dirichlet or Neumann eigenfunction, then*

$$\int_Z |\nabla \varphi|^2 dS \lesssim \lambda^{3/2}$$

Proof. This will follow from the identity

$$-\int_M \operatorname{sgn}(\varphi) \operatorname{div}(|\nabla \varphi| \nabla \varphi) dV = \int_{\partial M} |\partial_\nu \varphi|^2 dS + 2 \int_Z |\nabla \varphi|^2 dS$$

We first prove this identity. Note that $-\partial_\eta \varphi = |\nabla \varphi|$ over $Z \setminus \Sigma$. If φ is a Dirichlet eigenfunction, then we also have $|\nabla \varphi| = -\partial_\eta \varphi = |\partial_\nu \varphi|$ at any point on $\partial P \cap \partial M$ where η is defined. By the divergence theorem,

$$\begin{aligned} -\int_P \operatorname{div}(|\nabla \varphi| \nabla \varphi) dV &= -\int_{\partial P} |\nabla \varphi| \partial_\eta \varphi dS \\ &= \int_{\partial P \cap \partial M} |\partial_\nu \varphi|^2 dS + \int_Z |\nabla \varphi|^2 dS \end{aligned}$$

Similarly,

$$\int_N \operatorname{div}(|\nabla \varphi| \nabla \varphi) dV = \int_{\partial N \cap \partial M} |\partial_\nu \varphi|^2 dS + \int_Z |\nabla \varphi|^2 dS$$

Adding these equations establishes the identity. Now we have

$$\begin{aligned} \int_Z |\nabla \varphi|^2 dS &\leq \int_M \left| \operatorname{div}(|\nabla \varphi| \nabla \varphi) \right| dV \\ &\lesssim \|\varphi\|_{H^2(M)} \|\varphi\|_{H^1(M)} \\ &\lesssim \lambda^{3/2} \end{aligned}$$

\square

For a Dirichlet eigenfunction, we also need the following lemma.

Lemma 2.4. *If φ is a Dirichlet eigenfunction, then*

$$\left(\int_{\partial M} |\partial_\nu \varphi|^2 dS \right)^{1/2} \lesssim \lambda^{1/2}$$

This lemma follows from a much more general result obtained by Tataru [11]. There is also the following short proof.

Proof. Let X be a smooth first-order differential operator on M with $X = \partial_\nu$ over ∂M . Then, by Green's identity,

$$\begin{aligned} \int_M u[X, \Delta]u dV &= -\lambda \int_M uXu dV - \int_M u\Delta Xu dV \\ &= \int_M (\Delta u)(Xu) dV - \int_M u\Delta Xu dV \\ &= \int_{\partial M} (\partial_\nu u)(Xu) dS \\ &= \int_{\partial M} |\partial_\nu u|^2 dS \end{aligned}$$

Since $[X, \Delta]$ is a second-order differential operator, this yields

$$\begin{aligned} \int_{\partial M} |\partial_\nu u|^2 dS &= \int_M u[X, \Delta]u dV \\ &\lesssim \|u\|_{L^2(M)} \|u\|_{H^2(M)} \\ &\lesssim \lambda \end{aligned}$$

□

We can now prove Theorem 1.1.

Proof of Theorem 1.1. First assume φ is a Neumann eigenfunction. By Theorem 1.2 and Lemma 2.3,

$$\lambda \int_M |\varphi| dV = 2 \int_Z |\nabla \varphi| dS \lesssim \mathcal{H}(Z)^{1/2} \lambda^{3/4}$$

We can rewrite this as

$$\lambda^{1/2} \left(\int_M |\varphi| dV \right)^2 \lesssim \mathcal{H}(Z)$$

So by Lemma 2.2,

$$\lambda^{\frac{5-2n}{6}} \lesssim \mathcal{H}(Z)$$

Now assume φ is a Dirichlet eigenfunction. By Theorem 1.2, Lemma 2.3, and Lemma 2.4,

$$\lambda \int_M |\varphi| dV = \int_{\partial M} |\partial_\nu \varphi| dS + 2 \int_Z |\nabla \varphi| dS \lesssim \lambda^{1/2} + \mathcal{H}(Z)^{1/2} \lambda^{3/4}$$

We can rewrite this as

$$\lambda^{1/2} \left(\int_M |\varphi| dV \right)^2 \lesssim \mathcal{H}(Z) + \lambda^{-1/2}$$

Now applying Lemma 2.2 yields the desired estimates. □

Remark. If (2.2) is true, then we would have a better lower bound for the L^1 norm of φ . If φ is a Neumann eigenfunction, this would yield

$$\lambda^{\frac{8-3n}{12}} \lesssim \mathcal{H}(Z)$$

The same bound would hold if φ is a Dirichlet eigenfunction and $n \leq 4$.

REFERENCES

- [1] T. H. Colding and W. P. Minicozzi, II. *Lower bounds for nodal sets of eigenfunctions*, arXiv:1009.4156, to appear in Comm. Math. Phys.
- [2] H. Donnelly and C. Fefferman. *Nodal sets of eigenfunctions: Riemannian manifolds with boundary*. Analysis, et cetera, 251-262, Academic Press, Boston, MA, 1990.
- [3] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [4] D. Grieser. *L^p bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries*. Ph. D. Thesis. University of California, Los Angeles: USA, 1992.
- [5] R. Hardt and L. Simon. *Nodal sets for solutions of elliptic equations*. J. Differential Geom. 30 (1989), no. 2, 505-522.
- [6] H. Hezari and C. D. Sogge. *A natural lower bound for the size of nodal sets*, arXiv:1107.3440, preprint.
- [7] H. F. Smith. *Sharp $L^2 \rightarrow L^q$ bounds on spectral projectors for low regularity metrics*. Math. Res. Lett. 13 (2006), no 5-6, 967-974.
- [8] H. F. Smith and C. D. Sogge. *On the critical semilinear wave equation outside convex obstacles*. J. Amer. Math. Soc. 8 (1995), no. 4, 879-916.
- [9] H. F. Smith and C. D. Sogge. *On the L^p norm of spectral clusters for compact manifolds with boundary*. Acta Math. 198 (2007), 107-153.
- [10] C. D. Sogge and S. Zelditch. *Lower bounds on the Hausdorff measure of nodal sets*, Math. Res. Lett. 18 (2011), no. 1, 25-37.
- [11] D. Tataru. *On the regularity of boundary traces for the wave equation*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26 (1998), no. 1, 185-206.